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A greedy algorithm for some classes of integer programs☆

V.V. Shenmaier

Sobolev Institute of Mathematics, Pr. Koptyuga 4, Novosibirsk 630090, Russia

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Abstract

We establish a necessary and sufficient condition for a greedy algorithm to find an optimal solution in the case of integer programs with separable concave objective functions. This extends some well-known results for spanning trees, matroids, and greedoids. As a corollary we obtain one new generalization of matroids and integer polymatroids preserving the optimality of greedy solutions.

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1. Introduction

We consider the following problem. Suppose I is a finite nonempty set and $N = \{0, 1, 2, \dots, \}$. For any vector X, denote by $X(i)$ the ith coordinate of X. Let f be any separable concave function over N^I that is

$$f(X) = \sum_{i \in I} f_i(X(i)), \tag{1}$$

where all f_i are concave.

As an example we have a linear function: $f(X) = \sum_{i \in I} w(i)X(i)$, where $w \in R^I$. For boolean vectors, any separable function is some linear function plus some constant. So in this case, optimizing a separable function is equivalent to optimizing a linear one.

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E-mail addresses: shenmaier@mail.ru (V.V. Shenmaier).

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Suppose $\mathfrak{I} \subset N^I$ is any finite family of integer nonnegative vectors and $\mathscr{B}(\mathfrak{I})$ is the set of maximal vectors of $\mathfrak{I}$. The problem considered here is

$$\begin{aligned} &\text{maximize} && f(X) \\ &\text{subject to} && X \in \mathscr{B}(\mathfrak{I}). \end{aligned} \tag{2}$$

We can say that this is a metaproblem since a lot of problems in combinatorial optimization can be represented in this way. An equivalent problem is

to minimize a separable *convex* function over $\mathscr{B}(\mathfrak{I})$.

Notation. Suppose $i \in I$, $X \in N^I$ and $A \subseteq I$. Then, e_i is the vector with 1 at the ith position and 0 at other ones; $\mathbf{0}$ is the zero vector; $|X|$ is the component sum of X; $X(A) = \sum_{i \in A} X(i)$; $I(X) = \{i \in I \mid X(i) > 0\}$; $J(X) = \{i \in I \mid X + e_i \in \mathfrak{I}\}$.

1.1. A greedy algorithm

For solving problems (1), (2) we use a greedy algorithm. The algorithm works as follows:

at the beginning $X = \mathbf{0}$;

until $J(X) \neq \emptyset$, the algorithm replaces X by $X + e_i$, where i is such that $f(X + e_i) = \max_{j \in J(X)} f(X + e_j)$;

if $J(X) = \emptyset$, the algorithm stops and X is a *greedy solution*.

Remark. An element i such that $f(X + e_i) = \max_{j \in J(X)} f(X + e_j)$ can be not unique. So we can have more than one greedy solutions.

1.2. Our interests

The greedy algorithm seems to be very simple and fast. However, a greedy solution is often not optimal. We say that the greedy algorithm *finds an optimal solution* if for a given family $\mathfrak{I}$ and for any separable concave objective function f, the integer program (1), (2) is solvable by the greedy algorithm.

Our interests are in finding families $\mathfrak{I}$ such that the greedy algorithm finds an optimal solution.

2. Known results

Let us give some definitions. A family $\mathfrak{I} \subset N^I$ is called *down-monotone* if $\mathfrak{I}$ is nonempty and $X \leqslant Y \in \mathfrak{I}$ implies $X \in \mathfrak{I}$ whenever $X \in N^I$. A family $\mathfrak{I}$ is called an *integer polymatroid* if $\mathfrak{I}$ is finite, down-monotone, and the following property holds:

$$\text{if } X, Y \in \mathfrak{I} \text{ and } |X| < |Y|, \quad \text{then } J(X) \cap I(Y - X) \neq \emptyset. \tag{3}$$

An equivalent definition of integer polymatroids is the following. Any finite down-monotone family $\mathfrak{I} \subset N^I$ is an integer polymatroid if, for any vector $X \in N^I$, all maximal vectors of the family $\{Y \in \mathfrak{I} \mid Y \leqslant X\}$ have the same component sum. Boolean integer polymatroids are called *matroids* (in this paper we identify families of sets and families of boolean vectors). Rado and Edmonds established the following theorem.

Theorem 1 (Edmonds [3], Rado [9]). *Suppose a family $\mathfrak{I} \subset N^I$ is boolean and down-monotone. Then the greedy algorithm finds an optimal solution if and only if $\mathfrak{I}$ is a matroid.*

Proof. Rado–Edmonds's theorem concerns linear objective functions. But as we noted above, in the boolean case, optimizing linear functions and optimizing separable ones is the same.

Glebov extended Rado–Edmonds's result to the case of optimizing separable concave functions over any down-monotone family $\mathfrak{I}$.

Theorem 2 (Glebov [5]). *Suppose a family $\mathfrak{I} \subset N^I$* is *down-monotone*. *Then the greedy algorithm finds an optimal solution if and only if $\mathfrak{I}$ is an integer polymatroid.*

Similar results for the case of linear programs were obtained by Edmonds [2].

Example. *The spanning tree problem*. Suppose G is any connected undirected graph, E is the edge set of G, and the family $\mathfrak{I} \subset N^E$ consists of boolean vectors X such that $I(X)$ is some acyclic subgraph of G. Then $\mathfrak{I}$ is a matroid and the greedy algorithm is essentially the famous Kruskal's algorithm for finding a spanning tree of the maximum weight.

If $\mathfrak{I}$ is not down-monotone, conditions for the greedy algorithm to find an optimal solution are more complicated. We say that a family $\mathfrak{I} \subset N^I$ is an *accessible family* if $\mathfrak{I}$ is finite, nonempty, and the following conditions hold:

$$\text{if } X \in \mathfrak{I} \setminus \{\mathbf{0}\}, \quad \text{then } X - e_i \in \mathfrak{I} \text{ for some } i \in I; \tag{4}$$

$$\text{if } X \in \mathfrak{I} \setminus \mathscr{B}(\mathfrak{I}), \quad \text{then } X + e_j \in \mathfrak{I} \text{ for some } j \in I. \tag{5}$$

Goecke, Korte, and Lovasz established the following theorem.

Theorem 3 (Goecke et al. [6]). *Suppose $\mathfrak{I} \subset N^I$ is any boolean accessible family. Then the greedy algorithm finds an optimal solution if and only if the strong exchange property holds*

$$\textit{if } X \leqslant Y, \quad X \in \mathfrak{I}, \quad Y \in \mathscr{B}(\mathfrak{I}), \quad \textit{and} \quad i \in J(X) \setminus I(Y),$$

$$\textit{then } Y + e_i - e_j \in \mathfrak{I} \quad \textit{for some } j \in J(X) \cap I(Y) \setminus I(X). \tag{6}$$

The strong exchange property was introduced by Goetchel [7]. He proved the previous theorem for the case of *greedoids*, boolean accessible families satisfying condition (3).

More recent results on greedily optimizable integer programs were obtained in [1,4,8].

3. A new result

Define the *accessible kernel of a family* $\mathfrak{I} \subseteq N^I$ as $\mathscr{A}(\mathfrak{I}) = \{e_{i(1)} + \cdots + e_{i(k)} \mid e_{i(1)} + \cdots + e_{i(s)} \in \mathfrak{I},\ 1 \leqslant s \leqslant k\} \cup \{\mathbf{0}\}$.

Theorem 4. *Suppose $\mathfrak{I} \subset N^I$ is any finite nonempty family of vectors. Then the greedy algorithm finds an optimal solution if and only if the following conditions hold*:

$$\text{if } X \in \mathscr{A}(\mathfrak{I}) \setminus \mathscr{B}(\mathfrak{I}), \quad \text{then } J(X) \neq \emptyset; \tag{7}$$

$$\begin{aligned} &\text{if } X \leqslant Y, \quad X \in \mathscr{A}(\mathfrak{I}), \quad Y \in \mathscr{B}(\mathfrak{I}), \quad \text{and} \quad i \in J(X) \setminus I(Y - X), \\ &\quad \text{then } Y + e_i - e_j \in \mathfrak{I} \quad \text{for some } j \in J(X) \cap I(Y - X). \end{aligned} \tag{8}$$

To prove this theorem, we need two lemmas.

Lemma 1. *Suppose the greedy algorithm finds an optimal solution. Then the following properties hold*:

(a) $\mathscr{B}(\mathfrak{I}) \subseteq \mathscr{A}(\mathfrak{I})$;
(b) *all maximal vectors of* $\mathfrak{I}$ *have the same component sum*;
(c) *if* $X \in \mathscr{A}(\mathfrak{I}) \setminus \mathscr{B}(\mathfrak{I})$ *and* $Y \in \mathscr{B}(\mathfrak{I})$, *then* $J(X) \cap I(Y - X) \neq \emptyset$.

Proof. (a) Let $X \in \mathscr{B}(\mathfrak{I})$. Suppose $f_i(n) = \min\{n, X(i)\}$ ($n \in N$, $i \in I$) and $Y \in \mathscr{A}(\mathfrak{I})$ is a greedy solution. Then $f(Y) \geqslant f(X)$, which implies $Y \geqslant X$. But X is a maximal vector of $\mathfrak{I}$, hence $Y = X$, and so $X \in \mathscr{A}(\mathfrak{I})$.

(b) Set $f_i(n) = n$ ($n \in N$, $i \in I$). Using statement (a), we now obtain that any vectors $X, Y \in \mathscr{B}(\mathfrak{I})$ are greedy solutions. So $f(X) = f(Y)$, which implies $|X| = |Y|$.

(c) Suppose $X \in \mathscr{A}(\mathfrak{I}) \setminus \mathscr{B}(\mathfrak{I})$ and $Y \in \mathscr{B}(\mathfrak{I})$. Let $f_i(n) = \min\{n, \max\{X(i), Y(i)\}\}$ ($n \in N$, $i \in I$). Then X can be constructed after some steps of the greedy algorithm. Since $X \notin \mathscr{B}(\mathfrak{I})$, X is not a greedy solution. So there exists some sequence $i(1), \ldots, i(k) \in I$ ($k \geqslant 1$) such that the vector $X' = X + e_{i(1)} + \cdots + e_{i(k)}$ is a greedy solution and $X + e_{i(1)} + \cdots + e_{i(s)} \in \mathfrak{I}$ ($s = 1, \ldots, k$). Then $X' \in \mathscr{B}(\mathfrak{I})$ and $f(X') \geqslant f(Y)$. On the other hand, $f(X') \leqslant |X'|$ and $f(Y) = |Y|$. So $|X'| = |Y| = f(Y) = f(X')$ by statement (b). This yields $f(X + e_{i(1)} + \cdots + e_{i(s)}) = |X| + s$ ($s = 1, \ldots, k$) and so $f(X + e_{i(1)}) = f(X) + 1$. By the choice of f, we conclude that $i(1) \in I(Y - X)$. Thus $J(X) \cap I(Y - X) \neq \emptyset$, as required. □

Lemma 2. *Under conditions* (7), (8), *all maximal vectors of* $\mathfrak{I}$ *have the same component sum.*

Proof. First we prove that $\mathscr{B}(\mathfrak{I}) \subseteq \mathscr{A}(\mathfrak{I})$. Suppose $Y \in \mathscr{B}(\mathfrak{I}) \setminus \mathscr{A}(\mathfrak{I})$. Let X be a maximal vector such that $X \in \mathscr{A}(\mathfrak{I})$ and $X \leqslant Y$ (such a vector exists since $\mathbf{0} \in \mathscr{A}(\mathfrak{I})$). Then $X \neq Y$, and $X \notin \mathscr{B}(\mathfrak{I})$. From (7) it follows that $J(X) \neq \emptyset$, and so $J(X) \setminus$

$I(Y-X) \neq \emptyset$ by the choice of X. From this and (8) we obtain $J(X) \cap I(Y-X) \neq \emptyset$, which contradicts the choice of X. Hence, $\mathscr{B}(\mathfrak{I}) \subseteq \mathscr{A}(\mathfrak{I})$.

Let $X \in \mathscr{B}(\mathfrak{I})$. Then $X \in \mathscr{A}(\mathfrak{I})$, and so X can be represented as $X = e_{i(1)} + \cdots + e_{i(l)}$, where vectors $X_s = e_{i(1)} + \cdots + e_{i(s)}$ belong to $\mathfrak{I}$ $(s = 1, \dots l)$. Suppose now that $Y \in \mathscr{B}(\mathfrak{I})$, $|Y| > |X|$, and among all such vectors, Y defines the maximal number s such that $X_s \leqslant Y$, $X_{s+1} \nleqslant Y$. Then $s < l$ and $i(s+1) \in J(X_s) \setminus I(Y - X_s)$. From (8) it follows that $Y + e_{i(s+1)} - e_j \in \mathfrak{I}$ for some $j \in J(X_s) \cap I(Y - X_s)$. Let $Y' \in \mathscr{B}(\mathfrak{I})$, where $Y' \geqslant Y + e_{i(s+1)} - e_j$. Then $|Y'| > |X|$ and $X_{s+1} \leqslant Y'$, which contradicts the choice of Y. The lemma is proved. □

Proof of Theorem 4. Necessity.

Suppose $X \in \mathscr{A}(\mathfrak{I}) \setminus \mathscr{B}(\mathfrak{I})$. Then $J(X) \neq \emptyset$ by statement (c) of Lemma 1. Condition (7) is proved.

Suppose $X \leqslant Y$, $X \in \mathscr{A}(\mathfrak{I})$, $Y \in \mathscr{B}(\mathfrak{I})$, and $i \in J(X) \setminus I(Y - X)$. By statement (b) of Lemma 1, all maximal vectors of $\mathfrak{I}$ have the same component sum, say l. Now we construct the following sequence of vectors: Let $s = |X| + 1$ and $X_s = X + e_i$. Suppose X_s has been already defined $(s = |X| + 1, \dots, l-1)$, $X_s \in \mathscr{A}(\mathfrak{I})$, $|X_s| = s$. Hence, $X_s \notin \mathscr{B}(\mathfrak{I})$ and $J(X_s) \cap I(Y - X_s) \neq \emptyset$ by statements (b), (c) of Lemma 1. Then we can choose an element $i(s+1) \in J(X_s)$ such that

$$i(s+1) \in I(Y - X_s) \quad \text{and}$$

$$i(s+1) \notin J(X), \quad \text{whenever } J(X_s) \cap I(Y - X_s) \nsubseteq J(X).$$

Define $X_{s+1} = X_s + e_{i(s+1)}$. Then $|X_l| = l$, $X + e_i \leqslant X_l \leqslant Y + e_i$, and $X_l \in \mathscr{B}(\mathfrak{I})$. So $X_l = Y + e_i - e_j$ for some $j \in I$. Since $i \notin I(Y - X)$, we have $Y(i) = X(i) < X_l(i)$. Hence $Y(j) > X_l(j)$, and then $j \in I(Y - X)$. Thus, it remains to prove that $j \in J(X)$. Set

$$f_k(n) = \begin{cases} \min\{n, X(k)\}, & n \in N,\ k \in J(X), \\ \min\{n, Y(k)\}, & n \in N,\ k \in I \setminus J(X). \end{cases}$$

Then $f(Y) = l - (Y - X)(J(X))$ and $f(X_l) = l - (Y + e_i - e_j - X)(J(X))$. Since $i \in J(X)$, we have $f(X_l) = l - 1 - (Y - e_j - X)(J(X))$. By the construction, X_l is a greedy solution. It follows that $f(Y) \leqslant f(X_l)$, and so $(Y - X)(J(X)) \geqslant 1 + (Y - e_j - X)(J(X))$. Therefore, we obtain $j \in J(X)$.

Sufficiency. Suppose X is a greedy solution and X_s is constructed by the greedy algorithm after the step $s = 1, \dots, l$ $(X = X_l)$. From (7) we obtain $X \in \mathscr{B}(\mathfrak{I})$. Now suppose that $Y \in \mathscr{B}(\mathfrak{I})$, $f(Y) > f(X)$, and among all such vectors, Y defines the maximal number s such that $X_s \leqslant Y$, $X_{s+1} \nleqslant Y$. Then $X_{s+1} - X_s = e_i$ for some $i \in J(X_s) \setminus I(Y - X_s)$. Hence, $Y + e_i - e_j \in \mathfrak{I}$ for some $j \in J(X_s) \cap I(Y - X_s)$ by (8). Denote by Y' the vector $Y + e_i - e_j$. Then $Y' \in \mathscr{B}(\mathfrak{I})$ by Lemma 2. Also, we have $X_{s+1} \leqslant Y'$.

By the rules of the greedy algorithm, $f(X_s + e_i) \geqslant f(X_s + e_j)$. Set $n = X_s(i)$, $m = X_s(j)$, and $m' = Y'(j)$. Then

$$f_i(n+1) + f_j(m) \geqslant f_i(n) + f_j(m+1).$$

On the other hand, since f_j is concave and $m' \geqslant m$, we have

$$f_j(m'+1) - f_j(m') \leqslant f_j(m+1) - f_j(m).$$

It follows that $f_i(n+1) + f_j(m') \geqslant f_i(n) + f_j(m'+1)$. Since $i \notin I(Y - X_s)$, we obtain $n = Y(i)$ and $n+1 = Y'(i)$. Also, $m' = Y'(j)$ and $m'+1 = Y(j)$. Hence $f(Y') \geqslant f(Y)$, which contradicts the choice of Y. This completes the proof. □

4. A generalization of matroids

Using Theorem 4, we can get some new interesting integer programs solvable by the greedy algorithm. One of them is shown below.

First we introduce one definition. Suppose $\mathfrak{I} \subset N^I$ and a function r over N^I is such that $r(X) = \max\{|A| \mid A \in \mathscr{A}(\mathfrak{I}),\ A \leqslant X\}$. Then we say that r is the *rank function of* $\mathfrak{I}$.

In the case of matroids and integer polymatroids, this function has the following properties:

(r1) $r(\mathbf{0}) = 0$;
(r2) $r(X + e_i) - r(X) \in \{0, 1\}$;
(r3) $r(X + e_i) - r(X) \geqslant r(Y + e_i) - r(Y)$ whenever $X \leqslant Y$.

On the other hand, any integer polymatroid can be represented as a family $\{X \in N^I \mid r(X) \geqslant |X|\}$ for some function r with properties (r1)–(r3).

Suppose a function p over N^I is such that

(p1) $p(\mathbf{0}) \geqslant 0$;
(p2) $p(X + e_i) - p(X) \in \{\alpha_{|X|}, \beta_{|X|}\}$, where $\alpha_s < 0 \leqslant \beta_s$ $(s \in N)$;
(p3) $p(X + e_i) \geqslant p(X)$ whenever $p(Y + e_i) \geqslant p(Y)$ and $X \leqslant Y$.

Suppose $C \in N^I$ and a family $\mathfrak{I}$ consists of vectors $X \in N^I$ such that

$$p(X) \geqslant 0; \tag{9}$$

$$X \leqslant C. \tag{10}$$

Note that a function $r'(X) = r(X) - |X|$ satisfies (p1)–(p3) (for $\alpha_s \equiv -1$, $\beta_s \equiv 0$) whenever r satisfies (r1)–(r3). So any integer polymatroid can be represented in the form (9)–(10). In the general case, $\mathfrak{I}$ even need not be down-monotone.

Theorem 5. *Suppose a family* $\mathfrak{I}$ *is given by* (9), (10) *and* p *satisfies* (p1)–(p3). *Then the greedy algorithm finds an optimal solution.*

To prove this theorem, we use some auxiliary statements. Define

$$\mathfrak{I}_0 = \{X \leqslant C \mid p(X) \geqslant p(X - e_i) \text{ for all } i \in I(X)\}.$$

Then $\mathfrak{I}_0 \subseteq \mathfrak{I}$ since $p(\mathbf{0}) \geqslant 0$. For any vectors $X, A \in N^I$, we will say that A is a *base of* X if A is a maximal vector such that $A \in \mathfrak{I}_0$ and $A \leqslant X$.

Lemma 3. *$\mathfrak{I}_0$ is down-monotone.*

Proof. Let $X \leqslant Y \in \mathfrak{I}_0$ and $X \in N^I$. Then for any $i \in I(X)$, $p(Y) \geqslant p(Y - e_i)$, and so $p(X) \geqslant p(X - e_i)$ by (p3). Hence, $X \in \mathfrak{I}_0$. $\square$

Lemma 4. *Suppose $X \in N^I$. Then $X \in \mathfrak{I}_0$ if and only if $p(X) = p(\mathbf{0}) + \sum_{s=0}^{|X|-1} \beta_s$.*

Proof. Let $X \in \mathfrak{I}_0$, $X = e_{i(1)} + \cdots + e_{i(k)}$, $k = |X|$, and $X_s = e_{i(1)} + \cdots + e_{i(s)}$ $(s = 0, \ldots, k)$. By Lemma 3, $X_s \in \mathfrak{I}_0$ $(s \leqslant k)$. From this and (p2) it follows that $p(X_{s+1}) - p(X_s) = \beta_s$ $(s < k)$. Hence, $p(X) = p(\mathbf{0}) + \sum_{s=0}^{k-1} \beta_s$.

Let $X \notin \mathfrak{I}_0$. Then $p(X) < p(X - e_i)$ for some $i \in I(X)$. Suppose $X = e_{i(1)} + \cdots + e_{i(k)}$, $k = |X|$, $i(k) = i$, and $X_s = e_{i(1)} + \cdots + e_{i(s)}$ $(s = 0, \ldots, k)$. Then $p(X_{s+1}) - p(X_s) \leqslant \beta_s$ $(s < k)$ by (p2) and $p(X_k) - p(X_{k-1}) < 0 \leqslant \beta_{k-1}$. So $p(X) < p(\mathbf{0}) + \sum_{s=0}^{k-1} \beta_s$. The lemma is proved. $\square$

Lemma 5. *Suppose $X \in N^I$ and A is a base of X. Then,*

$$p(X) = p(\mathbf{0}) + \sum_{s=0}^{|A|-1} \beta_s + \sum_{s=|A|}^{|X|-1} \alpha_s.$$

Proof. Let $X = e_{i(1)} + \cdots + e_{i(k)}$, $k = |X|$, $X_s = e_{i(1)} + \cdots + e_{i(s)}$, and $X_{|A|} = A$. Suppose $|A| \leqslant s < k$ and $p(X_{s+1}) - p(X_s) \geqslant 0$. Then $p(A + e_{i(s+1)}) - p(A) = \beta_{|A|}$ by (p3), (p2). From this and Lemma 4 we obtain $A + e_{i(s+1)} \in \mathfrak{I}_0$, which contradicts the choice of A. So $p(X_{s+1}) - p(X_s) = \alpha_s$ by (p2). It follows that $p(X) = p(A) + \sum_{s=|A|}^{k-1} \alpha_s = p(\mathbf{0}) + \sum_{s=0}^{|A|-1} \beta_s + \sum_{s=|A|}^{k-1} \alpha_s$ by Lemma 4, as required. $\square$

Lemma 6. *$\mathfrak{I}_0$ is an integer polymatroid.*

Proof. By (10) and Lemma 3, $\mathfrak{I}_0$ is finite and down-monotone. Let $X \in N^I$ and A, B be two bases of X. From Lemma 5 it follows that $\sum_{s=0}^{|A|-1} \beta_s + \sum_{s=|A|}^{|X|-1} \alpha_s = \sum_{s=0}^{|B|-1} \beta_s + \sum_{s=|B|}^{|X|-1} \alpha_s$. Then $|A| = |B|$ since $\beta_s > \alpha_s$. This completes the proof. $\square$

Lemma 7. *Suppose $X \in N^I$, A is a base of X, and $i \in I$. Then $A + e_i \in \mathfrak{I}_0$ if and only if $p(X + e_i) \geqslant p(X)$.*

Proof. Let $A + e_i \in \mathfrak{I}_0$. Then $A + e_i$ is a base of $X + e_i$. Lemma 5 now yields $p(X + e_i) - p(X) = \alpha_{|X|} + \beta_{|A|} - \alpha_{|A|} > \alpha_{|X|}$ since $\beta_s > \alpha_s$. So $p(X + e_i) \geqslant p(X)$ by (p2).

Let $p(X + e_i) \geqslant p(X)$. Then $p(A + e_i) \geqslant p(A)$ by (p3). From this, (p2), and Lemma 4 it follows that $A + e_i \in \mathfrak{I}_0$, as required. $\square$

Lemma 8. *$J(X) \cap I(Y - X) \neq \emptyset$ whenever $X, Y \in \mathfrak{I}$ and $|X| < |Y|$.*

Proof. Let A be a base of X and B be a base of Y.

Case 1: $|B| > |A|$. Since $\mathfrak{I}_0$ is an integer polymatroid, we have $A + e_i \in \mathfrak{I}_0$ for some $i \in I(B - A)$. The choice of A implies $A \leqslant X$ and $A + e_i \nleqslant X$. Then $A(i) = X(i)$, and so $i \in I(Y - X)$. Also, $p(X + e_i) \geqslant 0$ since $p(X + e_i) \geqslant p(X)$ by Lemma 7.

Case 2: $|B| \leqslant |A|$. Suppose $i \in I(Y - X)$ and $p(X + e_i) < 0$. Then, using Lemma 5 and the relations $\alpha_s < 0 \leqslant \beta_s$, we obtain $p(Y) \leqslant p(0) + \sum_{s=0}^{|A|-1} \beta_s + \sum_{s=|A|}^{|Y|-1} \alpha_s \leqslant p(X + e_i) + \sum_{s=|X|+1}^{|Y|-1} \alpha_s \leqslant p(X + e_i) < 0$. This contradicts $Y \in \mathfrak{I}$, which implies $p(X + e_i) \geqslant 0$.

In both cases, we also have $X + e_i \leqslant Y \leqslant C$, and so $X + e_i \in \mathfrak{I}$. Hence $J(X) \cap I(Y - X) \neq \emptyset$, as required. □

Proof of Theorem 5. By (10), $\mathfrak{I}$ is finite. Relation (p1) yields $\mathbf{0} \in \mathfrak{I}$. From this and Lemma 8 it follows that condition (7) of Theorem 4 holds and $\mathfrak{I} = \mathscr{A}(\mathfrak{I})$. It remains to verify (8). Suppose $X \leqslant Y$, $X \in \mathfrak{I}$, $Y \in \mathscr{B}(\mathfrak{I})$, and $i \in J(X) \setminus I(Y - X)$. Then $X(i) = Y(i)$ and $X + e_i \in \mathfrak{I}$. Hence, $X \notin \mathscr{B}(\mathfrak{I})$ and $|X| < |Y|$.

Note that for any $j \in I(Y - X)$, the vectors $Y + e_i - e_j$ and $X + e_j$ satisfy (10) since $(Y + e_i - e_j)(i) = (X + e_i)(i) \leqslant C(i)$ and $(X + e_j)(j) \leqslant Y(j) \leqslant C(j)$. Then to verify (8), it suffices to find an element $j \in I(Y - X)$ such that $Y + e_i - e_j$ and $X + e_j$ satisfy (9) only.

Let A be a base of X. Since $\mathfrak{I}_0$ is an integer polymatroid, there exists $B \geqslant A$, a base of Y.

Case 1: $A + e_i \in \mathfrak{I}_0$. Suppose $A = B$. Then from Lemma 7 it follows that $p(Y + e_i) \geqslant p(Y) \geqslant 0$. On the other hand, $(Y + e_i)(i) = (X + e_i)(i) \leqslant C(i)$ since $X + e_i \in \mathfrak{I}$. Thus $Y + e_i \in \mathfrak{I}$, which contradicts the choice of Y. Then $A \neq B$, $|B| > |A|$, and so $|B| \geqslant |A + e_i|$. From this and the property (3) of integer polymatroids, we can construct some vector $B' \in \mathfrak{I}_0$ such that $A + e_i \leqslant B' \leqslant B + e_i$ and $|B'| = |B|$. So $B' = B + e_i - e_j$ for some $j \in I$. By the choice of A, we have $A \leqslant X$ and $A + e_i \nleqslant X$. Then $A(i) = X(i) = Y(i)$, and so $A + e_i \nleqslant Y$, $A + e_i \nleqslant B$. Hence $B \neq B'$, $j \in I(B - B')$, and then $j \in I(B - A)$.

Since $\mathfrak{I}_0$ is down-monotone, we have $A + e_j \in \mathfrak{I}_0$. The choice of A implies $A \leqslant X$ and $A + e_j \nleqslant X$. Then $A(j) = X(j)$, and so $j \in I(Y - X)$.

By Lemma 7, we have $p(X + e_i) \geqslant p(X) \geqslant 0$. Since $Y + e_i - e_j \geqslant B'$, $|B'| = |B|$, and $\beta_s \geqslant \alpha_s$, we have $p(Y + e_i - e_j) \geqslant p(Y) \geqslant 0$ by Lemma 5. Thus, $Y + e_i - e_j$ and $X + e_j$ satisfy (9).

Case 2: $A + e_i \notin \mathfrak{I}_0$. Using Lemma 8, we can construct some vector $Y' \in \mathfrak{I}$ such that $X + e_i \leqslant Y' \leqslant Y + e_i$ and $|Y'| = |Y|$. So $Y' = Y + e_i - e_j$ for some $j \in I$. Since $X + e_i \nleqslant Y$, we have $Y' \neq Y$, $j \in I(Y - Y')$, and so $j \in I(Y - X)$. It remains to show that $X + e_j$ satisfies (9).

By Lemma 7 and (p2), we have $p(X + e_i) = p(X) + \alpha_{|X|}$. From this and (p2) it follows that $p(X + e_j) \geqslant p(X + e_i)$, which implies $p(X + e_j) \geqslant 0$. This completes the proof. □

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